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Dynamical scenarios of changes in asymmetric relationships over time (2)

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Abstract

This paper is the second part of the revised version of my paper presented elsewhere (Chino, 2016). In this paper we propose a recent version of a set of complex difference equation models which describes changes in asymmetric relationships among objects over time. Typical examples of these objects may be citation frequencies in scientific publications, amounts of trade among nations, connections among neurons, and so on. Supposing that such an asymmetric relational data matrix or a weighted digraph is observed in an instant of time, we first apply the Chino-Shiraiwa theorem to the matrix and embed objects in a (complex) Hilbert space. Then we shall apply our recent version of a set of general complex difference equation models to the initial configuration of objects embedded in the Hilbert space. As a result, we have various possible theoretical scenarios of the trajectories of objects in this space. In this paper, we show some of the possible scenarios of the nonlinear difference equation models in the special cases when the number of objects are two and three.

Key words: complex difference equation, Hilbert space, Chino-Shiraiwa theorem, dynamic weighted digraph, chaos, trade imbalance, neural network.

1. Introduction

In a companion paper, we have reported the first part of the revised version of my handout presented at the 45th annual meeting of the Behaviormetric Society of Japan (Chino, 2017), in which we have described a current difference equation model and have shown the possible theoretical scenarios of its trajectories with some special *linear difference equation models*. In this paper we shall discuss the possible theoretical scenarios with some *nonlinear difference equation models*, as a second part of the revised version.

As discussed in the companion paper, we assume the following nonlinear difference equation models which include the linear models:

$$\boldsymbol{z}_{j,n+1} = \boldsymbol{z}_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk}^{(m)} \boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}), \quad j = 1, 2, \cdots, N, \quad (1)$$

$$\boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}) = \left(\left(z_{j,n}^{(1)} - z_{k,n}^{(1)}\right)^m, \left(z_{j,n}^{(2)} - z_{k,n}^{(2)}\right)^m, \dots, \left(z_{j,n}^{(p)} - z_{k,n}^{(p)}\right)^m \right)^t, \quad (2)$$

and

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$$\boldsymbol{D}_{jk}^{(m)} = diag\left(w_{jk}^{(1,m)}, w_{jk}^{(2,m)}, \dots, w_{jk}^{(p,m)}\right), \tag{3}$$

where m = 1, 2, ⋯, q.

Here, $\boldsymbol{z}_{j,n}$ denotes the coordinate vector of member j at time n in a p-dimensional Hilbert space or an indefinite metric space. Moreover, m denotes the degree of the vector function, $\boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n})$, which is assumed to have the maximum value q, while $w_{jk}^{(1,m)}, w_{jk}^{(2,m)}, \dots, w_{jk}^{(p,m)}$ in Eq. (3) are *complex constants*. These models are very general and might enable us to describe various possible changes in asymmetric relationships among members over time.

2. Dynamical scenarios of changes in asymmetric relationships in complex difference equation models

In this section, first we consider the difference equation models with quadratic terms.

Here, we assume the special case when m=2, p=1, $\boldsymbol{g}(\boldsymbol{u}_{j,n})=0$, and $\boldsymbol{z}_0=0$ in Eqs. (1) to (3). Such a system is written as

$$\begin{cases} z_{j,n+1} = z_{jn} + \alpha_{jk}^{(1)}(z_{jn} - z_{kn}) + \alpha_{jk}^{(2)}(z_{jn} - z_{kn})^2, \\ z_{k,n+1} = z_{kn} + \alpha_{kj}^{(1)}(z_{kn} - z_{jn}) + \alpha_{kj}^{(2)}(z_{kn} - z_{jn})^2. \end{cases} \tag{4}$$

This type of system has a very desirable property in that we can utilize the heritage of the theory of the complex dynamical system developed in mathematics directly in classifying its trajectories. In fact, defining a new variable, $u_{jkn} = z_{jn} - z_{kn}$, and transforming it linearly, we have a new system

$$Z_{jk,n+1} = Z_{jk,n}^2 + \gamma_{jk}^{(1)}, \quad \text{where } \gamma_{jk}^{(1)} = \frac{1}{2}\beta_{jk}^{(1)} - \frac{1}{4}\left[\beta_{jk}^{(1)}\right]^2, \tag{5}$$

and $\beta_{jk}^{(1)} = 1 + \alpha_{jk}^{(1)} + \alpha_{kj}^{(1)}$ (K. Shiraiwa, personal communication, April 25, 2014). Depending on the value of $\gamma_{jk}^{(1)}$, we have the Mandelbrot set. It is interesting to note that $\beta_{jk}^{(1)}$, which determines the value of $\gamma_{jk}^{(1)}$, is one of the eigenvalues of the coefficient matrix of the linear system discussed in the companion paper. We will see later that even a quadratic system which is a special case of our general model described by Eqs. (1) to (3) exhibits excitingly richer dynamical properties of its solution curve than linear systems.

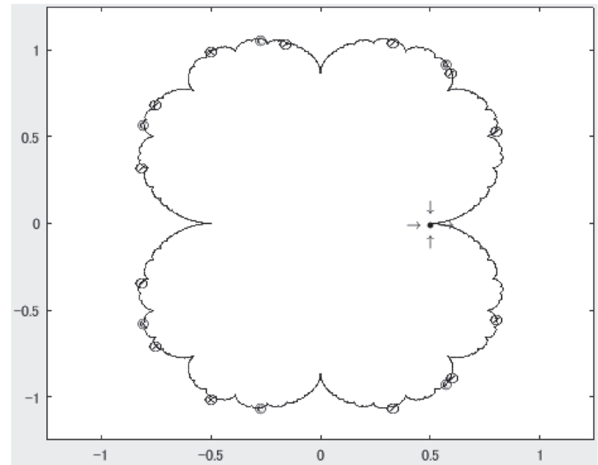

Fig. 1. Julia set for $Z_{jk,n+1} = Z_{jk,n}^2 + 0.25$ with a *parabolic fixed point* (0.5) with the '•' mark, two repelling periodic points of period 2 (−0.5±i) with the '⊗' mark, and six repelling periodic points of period 3 with the '©' mark, and twelve repelling periodic points of three 4-sycles with the '⊘' mark.

Before we discuss the dynamical scenario of this nonlinear system, we consider the system described by Eq. (5), which is naturally induced from Eq. (4). As pointed out earlier, the induced new system, $Z_{jk,n+1} = Z_{jk,n}^2 + \gamma_{jk}^{(1)}$, which is obtained by a linear transformation of the difference, $z_{jn} - z_{kn}$, is of the form assumed in discussing the famous Mandelbrot set. Depending on the complex constant term, $\gamma_{jk}^{(1)}$, there exist various dynamical scenarios of this new system, which are well known in the area of the *complex dynamical system* in mathematics (e.g., Milnor, 2000; Peitgen & Richter, 1986; Ueda et al., 1995).

In examining the dynamical scenarios of Z_{jkn}, we can utilize various theoretical results established in the complex dynamical system. Some of the key phrases in these results are the fixed point as well as the periodic point (orbit), the multiplier of these points, and the Fatou set and Julia set (e.g., Carleson & Gamelin, 1993).

Suppose f is a *holomorphic function*, that is, an analytic function in a complex space. Then, z_f is called a fixed point if $f(z_f)=z_f$. The number

$\lambda = f'(z_f)$ is called the multiplier of the fixed point. Here, f' is the first derivative of f with respect to z_f. The multiplier determines the property of the fixed point as follows (e.g., Carleson & Gamelin, 1993, p.27):

(1) Attracting if $|\lambda|<1$. (If $\lambda = 0$, we refer to a *superattracting* fixed point.)

(2) *Repelling* if $|\lambda|>1$.

(3) *Rationally neutral* if $|\lambda|=1$ and $\lambda^n=1$ for some integer n.

(4) *Irrationally neutral* if $|\lambda|=1$ but λ^n is never 1.

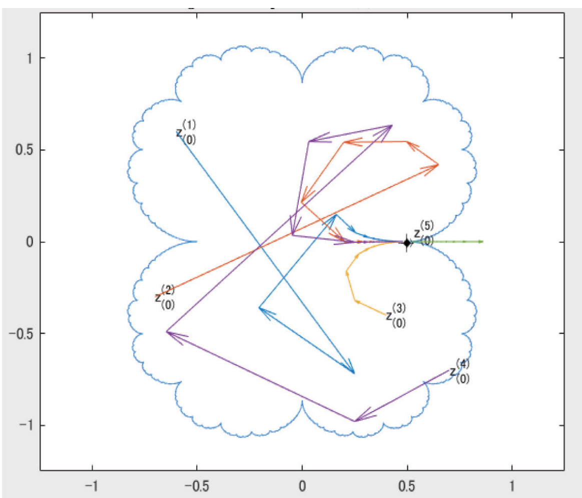

Fig. 2. Trajectories of the system with several initial points inside the Julia set.

In contrast, the *periodic point* (periodic *orbit*, or *cycle*) is defined as follows (e.g., Milnor, 2000, p.43). Let us suppose that we iterate the holomorphic map m times using the above function f. We shall denote this m iterates by $f^{\circ m}$. If $z_1, \cdots, z_m$ are all distinct, and if z_m is equal to z_0, then the integer $m \geq 1$ is called the *period*, that is,

$$f: z_0 \to z_1 \to z_2 \ldots \to z_{m-1} \to z_m = z_0. \qquad (6)$$

In this sense, the fixed point is a periodic orbit of period 1. We can say that a periodic point of period m is the fixed point of $f^{\circ m}$. Classification of the periodic orbit is the same as that of the fixed point discussed above (e.g., Beardon, 1991, p.91; Ueda et al., 1995, p.37). In this case, however, the *multiplier of the periodic point* is defined as $\lambda = [f^{\circ m}(z_p)]'$. By the Chain Rule, we have $\lambda = [f^{\circ m}(z_p)]' = \prod_{k=0}^{m-1} f'(z_k)$ (e.g., Beardon, 1991). The neutral point is sometimes called the *indifferent point*, and the rationally neutral point is sometimes called the *parabolic point*.

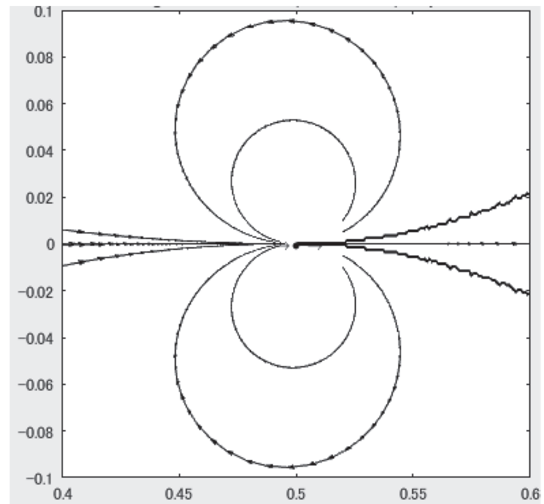

Fig. 3. Nearby trajectories which go in or out of the parabolic fixed point.

Let f be the holomorphic function on the complex plane, and let the divergent set of its collection of iterates be I_p. The complement of I_p is called the *filled-in Julia* set K_p, and the boundary of K_p is called the *Julia set*. The *Fatou set* is the complement of the Julia set. It is known that a point p_0 belongs to the Julia set if and only if dynamics in a neighborhood of p_0 displays "sensitive dependence on initial conditions" (e.g., Milnor, 2000, p.38).

It is well known that

(1) Every attracting periodic point is contained in the Fatou set of f.

(2) Every repelling periodic point is contained in the Julia set of f.

(3) Every rationally neutral (or parabolic) periodic point belongs to the Julia set of f,

(4) Every irrationally neutral periodic point includes *Siegel points* (the Fatou set) or *Cremer points* (the Julia set)

(e.g., Milnor , 2000, pp.43-44, pp.116-131; Ueda et al., 1995, p.12 for the Cremer points) .

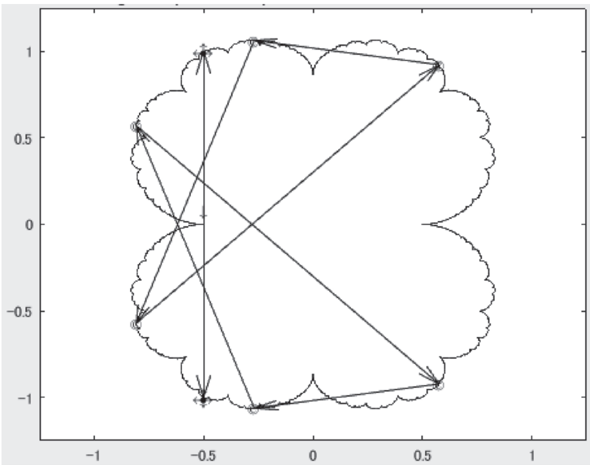

Fig. 4. Trajectories of the two repelling periodic points with period 2 ($-0.5\pm i$), and 6 repelling periodic points with period 3 which constitute two groups .

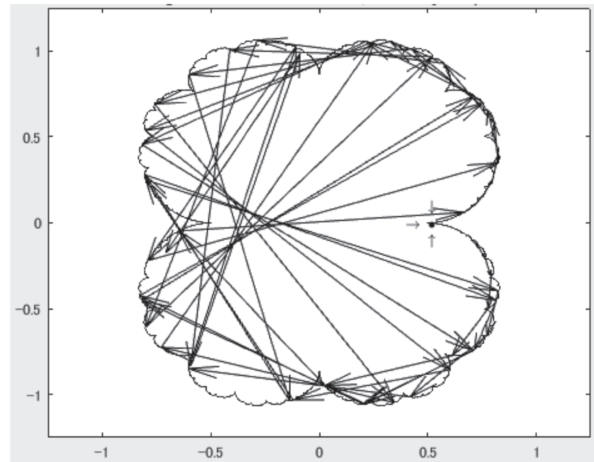

Fig. 6. A trajectory after 54 iterations, whose initial point is almost on the Julia set, that is, $0.700607338144023705-0.1i$.

It is also well known that if f maps the Fatou component U onto itself, then there are just four possibilities, as follows: Either U is the immediate basin for an attracting fixed point, or for one petal of a parabolic fixed point which has multiplier $\lambda = 1$, or else U is a Siegel disk or *Herman ring*. However, no polynomial map can have a Herman ring (e.g., Milnor, 2000, pp.152-153).

Regarding the number of the above special sets, it is known that there always exist *infinitely many repelling cycles*. Another important theorem is that a rational map can have at most *finitely many parabolic periodic points*. In fact for a map of degree d ≧ 2 the number of parabolic cycles plus the number of attracting cycles is at most $2d-2$ (Milnor, 2000, p.140, p.112).

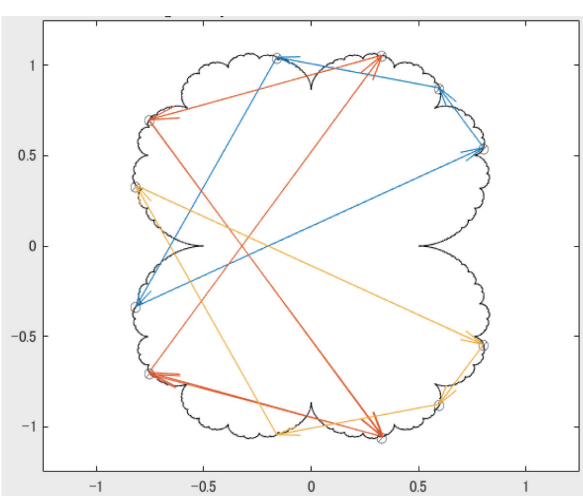

Fig.5. Trajectories of the three 4-cycles whose repelling periodic points are on the Julia set, i.e., the cauliflower set.

Hereafter, we will discuss the dynamical scenarios of Eq. (5) in two cases:

Simulation Case 1 is the case when $\gamma_{jk}^{(1)}=0.25$, that is, $Z_{jk,n+1}=Z_{jk,n}^2+0.25$.

This model is contained in the Mandelbrot set, and is known as the "*cauliflower set*".

Figure 1 shows the cauliflower set with a parabolic fixed point and several repelling parabolic points. It is clear that all of these repelling points as well as the parabolic point are contained in the cauliflower Julia set from the theorem on the relation between these points and the Julia set.

Figure 2 shows the trajectories of the system with several initial points. It is apparent that all the trajectories inside the Julia set except $z_{(0)}^{(5)}$ converges to the para-bolic fixed point (0.5) as time proceeds. To be precise, these curves are tangent to thereal axis. In contrast, $z_{(0)}^{(5)}$ which is located a bit to the right of the fixed point on the real axis diverges to infinity on the positive real axis.

Figure 3 zooms in the parabolic fix point to show trajectories going in or out of it. This is the very feature of the parabolic fixed point which is well

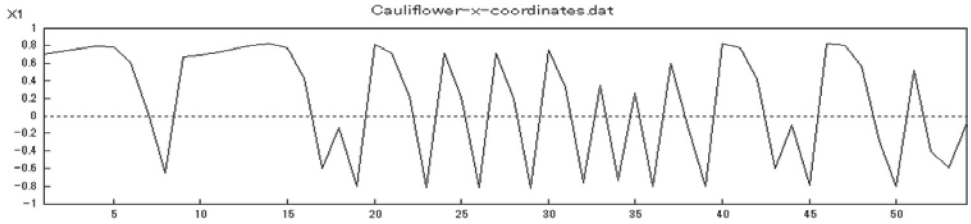

Fig. 7. Values of the real axis of the trajectory of the system whose initial value is almost on a Julia set (cauliflower set).

known elsewhere (e.g., Carleson & Gamelin, 1993, p.37).

Figure 4 shows trajectories of the two repelling periodic points with period 2 ($-0.5\pm i$), and six repelling periodic points with period 3. It is apparent that these six periodic points constitute two groups, one being in the upper part and the other in the relatively lower part of this figure, both of which constitute 3-cycles.

Figure 5 shows the three 4-cycles whose twelve repelling points are on the Julia set, that is, the cauliflower set.

Figure 6 is a trajectory after 54 iterations, whose initial point is *almost* on the Julia set, that is, $0.700607338144023705-0.1i$. We define such a point as the one just before the trajectory falls inside the Julia set visually, and is actually computed by a program with high precision like MATLAB.

Figure 7 depicts 54 values of the real axis of the trajectory of the system whose initial value is almost on a Julia set (cauliflower set). It is apparent that these points exhibit a random behavior at a glance, reflecting the sensitive dependence on initial conditions. Values of the imaginary axis also behave randomly like those of the real axis.

To sum up, defining a new variable, $u_{jkn} = z_{jn} - z_{kn}$, and considering a new system which is obtained by transforming u_{jkn} linearly, that is, $Z_{jk,n+1} = Z_{jk,n}^2 + 0.25$ (the cauliflower set), its solution curves can be classified as follows, depending on the initial value:

(1) In the case when the initial point is outside the Julia set (the cauliflower) :
The solution curves diverge to infinity.

(2) In the case when the initial point is on

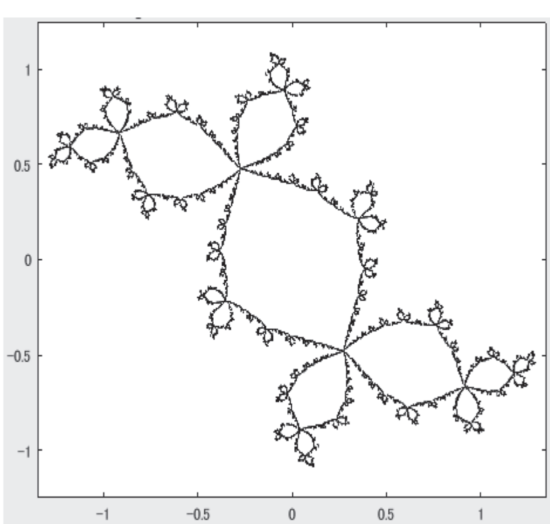

Fig. 8. Julia set for $Z_{jk,n+1} = Z_{jk,n}^2 -0.1200138225+0.7400145000i$.

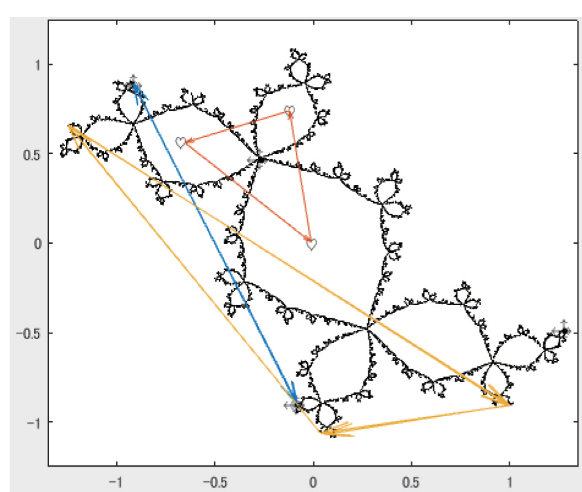

Fig. 9. Trajectories of the one repelling 2-cycle, one *attracting* 3-cycle, and one *repelling* 3-cycle, with two repelling fixed points.

the Julia set:

2-1) when it is on the parabolic fixed point (0.5):

The solution curves continue to stay in the fixed point.

2-2) when it is on the repelling periodic points with period 2 or 3:

The solution curves continue to go around the corresponding periodic points.

(In Figs. 4 to 5 we depicted up to 4-cycles. Theoretically, infinitely many repelling points exist.)

2-3) when it is on the Julia set excluding the parabolic fixed point and the repelling periodic points:

The solution curves continue to move irregularly on the Julia set.

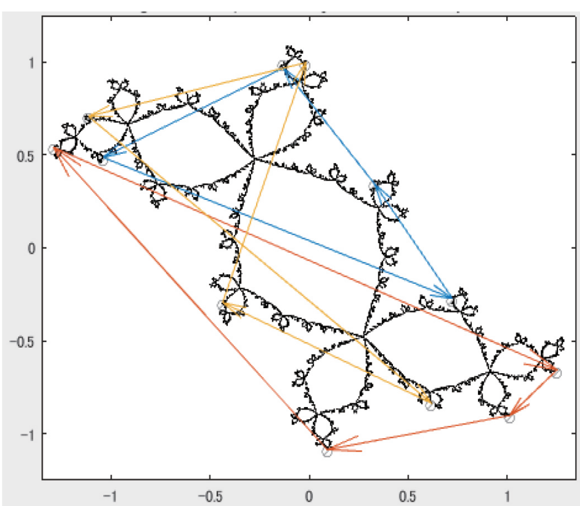

Fig. 10. Trajectories of the three repelling 4-cycles.

(3) In the case when the initial point is inside the Julia set:

The solution curves converge to the parabolic fixed point in a direction tangent to the real axis.

It should be noticed from Simulation Case 1 that *nonlinear systems* exhibit extremely richer behaviors of the solution curve than linear systems discussed earlier, even in a quadratic dyadic system. It should be noticed further that the above case is merely one of the examples of the Mandelbrot set. Depending on the parameter $\gamma_{jk}^{(1)}$ of Eq. (5), we encounter infinitely many dynamical scenarios of solution curves of this equation which is the transformed quadratic system of the original dyadic system, Eq. (4).

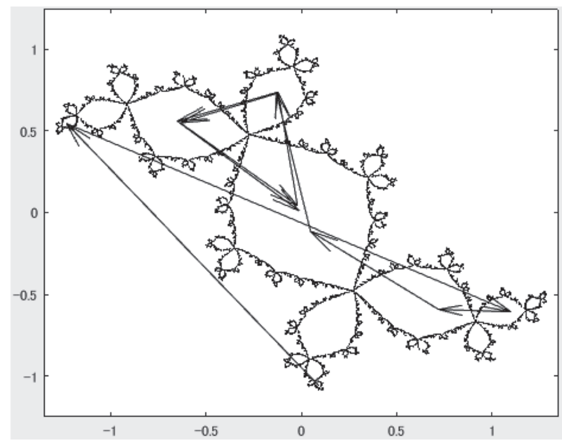

Fig. 11. Trajectory starting from an initial point in a small bud inside the Julia set, which is located almost at the bottom of this figure.

Simulation Case 2 is the case when $\gamma_{jk}^{(1)}$ = $-0.1200138225+0.7400145000i$, that is, $Z_{jk,n+1}$ = $Z_{jk,n}^2$ $-0.1200138225+0.7400145000i$ (Figure 8).

This system is almost the same as the case where $\gamma_{jk}^{(1)}$ $=-0.12+0.74i$, which is described by Peitgen and Richter (1986, p.10, Fig. 4). This system has two *repelling* fixed points, thus these points are on the Julia set. Moreover, it has one 2-cycle composed of two repelling periodic points. It has two 3-cycles, one of which is a 3-cycle composed of three repelling periodic points, and the other of which is a 3-cycle which is composed of three *attracting* periodic points. Figure 9 shows these points. Furthermore, it has three 4-cycles each of which are composed of four repelling periodic points. Figure 10 shows this. Figure 11 shows the trajectory starting from an initial point in a small bud inside the Julia set, which is located almost at the bottom of this figure.

3. Discussion

In this paper we have examined some of the possible scenarios of our nonlinear differ-rence equation models, especially those models on dyadic relations with a quadratic term described by Eq. (4) in section 2. These models are composed of two difference equations with a quadratic term in addition to linear terms, each of which describes the changes in the coordinate of the corresponding object in a Hilbert space over time.

It is easy to show that these two equations can be converted into a complex difference equation, that is, Eq. (5) in section 2, with a quadratic term and a complex constant by taking the difference between the coordinates of two objects. Since Eq. (5) includes a *Mandelbrot set* (e.g., Mandelbrot, 1977) as a special case (Chino, 2014) depending on the value $\gamma_{jk}^{(1)}$, this system exhibits various scenarios of its solution curves. As discussed in the previous section, we can utilize various theoretical results established in the *complex dynamical system* in examining these scenarios of Z_{jkn}.

We conducted a small simulation study in which two cases were examined. One is the case where $\gamma_{jk}^{(1)} = 0.25$ which constitutes the so-called *cauliflower* set, and the other is the case where $\gamma_{jk}^{(1)} = -0.1200138225+0.7400145000i$ which is frequently cited elsewhere (e.g., Peitgen & Richter, 1886). As shown in the simulation study, even in the dyadic case, possible scenarios of an aspect of the special cases of our nonlinear difference equation models are innumerable in contrast with the linear difference equation models which were discussed in a companion paper.

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